Analysis of a Living Fluid Continuum Model

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Dedicated to Professor Yoshikazu Giga on the occasion of his 60th birthday

Abstract Generalized Navier-Stokes equations which were proposed recently to describe active turbulence in living fluids are analyzed rigorously. Results on well-posedness and stability in the $L^2(\mathbb{R}^n)$-setting are derived. Due to the presence of a Swift-Hohenberg term global wellposedness in a strong setting for arbitrary initial data in $L^2_\sigma(\mathbb{R}^n)$ is available. Based on the existence of global strong solutions, results on linear and nonlinear (in-) stability for the disordered steady state and the manifold of ordered polar steady states are derived, depending on the involved parameters.

1 Introduction

There is a need to study analytical properties of generalized Navier-Stokes equations which were recently proposed [33, 8, 7] for active soft matter (for recent reviews see [25, 21, 22, 3]) to describe the dynamics of "living fluids" such as dense bacterial suspensions at low Reynolds number [24]. On a continuum scale, a living fluid flows with an internal speed that is set by the internal self-propulsion velocity of the bacteria. Generalized incompressible Navier-Stokes equations were designed to describe the spontaneous formation of fluid vortices on the mesoscale by including higher order derivatives in the velocities entering into the stress tensor. Indeed, at high

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density of the bacterial suspension, experiments reveal the spontaneous formation of meso-scale vortices [33], which is confirmed by particle-resolved simulations of self-propelled particles [34] and is consistent with the predictions of the Navier-Stokes equations generalized to living fluids. Therefore these continuum equations constitute an important general framework for flow of living or active fluids and provide a minimal continuum model for swirling. Though different to ordinary turbulence, which occurs at high Reynolds number, this phenomenon is often called "active turbulence" [37]. Active turbulence, which occurs at small Reynolds number, is characterized by scaling laws different to ordinary turbulence [33, 4].

Therefore a thorough mathematical study of this generalized Navier-Stokes system is highly desirable both from a physical and a mathematical point of view. This paper concerns the following minimal hydrodynamic model to describe the bacterial velocity in the case of highly concentrated bacterial suspensions with negligible density fluctuations considered on the domain $(0,\infty)\times\mathbb{R}^n$:

$$\begin{aligned} v_t + \lambda_0 v\cdot\nabla v &= f - \nabla p + \lambda_1 \nabla|v|^2 - (\alpha+\beta|v|^2)v + \Gamma_0\Delta v - \Gamma_2\Delta^2 v \\ \operatorname{div} v &= 0 \\ v(0) &= v_0 \end{aligned} \tag{1}$$

Here $v:(0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ is the (vectorial) bacterial velocity field and $p:(0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ the (scalar) pressure. The first equation is the conservation of momentum and the equation $\operatorname{div} v = 0$ results from conservation of mass and the assumption of constant density.

The generalized Navier-Stokes equations defined in (1) were originally proposed by Wensink et al. in [33] and then considered in Refs. [8, 7]. Clearly, for $\lambda_0 = 1$, $\lambda_1 = \beta = \Gamma_2 = 0$, and $\Gamma_0 > 0$, the model reduces to the incompressible Navier-Stokes equations in n spatial dimensions. Let us briefly discuss the physics behind the various terms entering in (1). The parameter λ_0 describes advection and nematic interactions and λ_1 is a prefactor in front of an active pressure contribution [7]. The two parameters λ_0 and λ_1 depend on the hydrodynamic nature of the swimmer, i.e. whether they are pushers or pullers [9], and on the dimension $n\in\{2,3\}$. The term involving the parameters α and β pushes the system towards rest with velocity $v=0$ if $\alpha>0$ and towards a characteristic non-vanishing velocity of $\sqrt{-\alpha/\beta}$ if $\alpha<0$ and $\beta>0$ as in the Toner-Tu model [31] corresponding to a quartic Landau-type velocity potential. If the parameter Γ_0 is positive, it determines the suspension's viscosity similar to the pure Navier-Stokes case. If it is negative, $\Gamma_2>0$ is required for stability reasons as in the traditional Swift-Hohenberg equation [30], for a review see [10]. By this fact, here we always assume $\Gamma_2>0$.

A main objective of this note is to provide an analytical approach to the generalized Navier-Stokes equations (1) in the $L^2(\mathbb{R}^n)$-setting. This will be performed in Section 3. There we will consider the general system (5), which includes the transformed systems about the steady states given in Section 2. These are the *disordered isotropic state* and the manifold of *globally ordered polar states*. The main results of Section 3 are as follows. Subsection 3.1 provides an approach to the linearized equations. The corresponding linear operator admits a bounded H^∞-calculus, see

Lemma 1. Propositions 1 and 2 give precise information on linear (in-) stability of the steady states depending on the values of the involved parameters. In Subsection 3.2 we will prove global (strong) wellposedness for the generalized Navier-Stokes equations (1) in the L^2-setting, see Theorem 2. Based on the global solvability, Subsection 3.3 concerns nonlinear (in-) stability. Theorem 3 transfers the linear stability results for the disordered state to the nonlinear situation. Theorem 4 then proves a nonlinear instability result for the ordered polar state.

We note that the physically relevant steady states (see Section 2) were given earlier in [33]. There also a linear instability analysis is performed on a non-rigorous level. Here we provide a rigorous analysis and go beyond linear instability.

The fact that we can prove the existence of a global unique strong solution for arbitrary initial data in $L^2_\sigma(\mathbb{R}^n)$ for system (1) of course is due to the presence of the Swift-Hohenberg term $\Gamma_2\Delta^2 u$. It causes the nonlinear terms to appear less strong compared to the classical second order Navier-Stokes equations. We refer to [11, 29, 35, 2, 5] and the references cited therein for more information on the classical Navier-Stokes equations.

We also remark that the purpose of this note is not to present best possible results in every direction. The L^2-approach given here is merely a first step towards a thorough analysis of the active fluids continuum model (1) in a variety of further significant situations. Further developments and future projects, e.g. including fluid boundaries, are addressed in Section 4.

2 Steady States

We assume $\Gamma_2, \beta > 0$ and $\alpha \in \mathbb{R}$. Then the following physically relevant stationary solutions appear [33]:

$$(v, p) = (0, p_0) \tag{2}$$

with a pressure constant p_0 and, if $\alpha < 0$, additionally

$$(v, p) = (V, p_0), \tag{3}$$

where $V \in B_{\alpha,\beta} := \{x \in \mathbb{R}^n : \; |V| = \sqrt{-\alpha/\beta}\}$, i.e., V denotes a constant vector with arbitrary orientation and fixed swimming speed $|V| = \sqrt{-\alpha/\beta}$.

The steady state (2) corresponds to a *disordered isotropic state* and (3) to the manifold $B_{\alpha,\beta}$ of *globally ordered polar states*.

Note that mathematically there is a further manifold of stationary solutions given by

$$v(x) = v_0, \qquad p(x) = p_0 - (\alpha + \beta|v_0|^2)v_0 \cdot x, \qquad x \in \Omega,\, p_0 \in \mathbb{R}, \tag{4}$$

with $v_0 \in \mathbb{R}^3$ arbitrary. For $v_0 = 0$ or $|v_0| = \sqrt{-\alpha/\beta}$ these solutions correspond to the above steady states (2) and (3), respectively. For all other values of v_0 they are, however, physically not relevant since their pressure takes arbitrary large negative values. Thus, in the sequel we will only consider (2) and (3).

3 Wellposedness and Stability

We perform an approach to the hydrodynamic model (1) in the $L^2(\mathbb{R}^n)$-setting. In order to include the steady states in our analysis we consider the following generalized system:

$$\begin{aligned} u_t + \lambda_0\left[(u+V)\cdot\nabla\right]u + (M+\beta|u|^2)u - \Gamma_0\Delta u + \Gamma_2\Delta^2 u + \nabla q &= f + N(u), \\ \operatorname{div} u &= 0, \\ u(0) &= u_0. \end{aligned} \tag{5}$$

Here $q = p - \lambda_1|v|^2$, $M \in \mathbb{R}^{n\times n}$ is a symmetric matrix, and $N(u) = \sum_{j,k} a_{jk}u^j u^k$ with $(a_{jk})_{j,k=1}^n \subset \mathbb{R}^n$ is a quadratic nonlinear term. By setting

$$V = 0, \quad M = \alpha, \quad N(u) = 0 \tag{6}$$

we obtain (1) for $u = v$, i.e., the system corresponding to the steady state (2) and by setting

$$V \in B_{\alpha,\beta}, \quad M = 2\beta VV^t, \quad N(u) = -\beta|u|^2 V - 2\beta(u\cdot V)u \tag{7}$$

we obtain the system for $u = v - V$ corresponding to (3). Note that for the appearing parameters we always assume that $\lambda_0, \lambda_1, \Gamma_0, \alpha \in \mathbb{R}$ and that $\Gamma_2, \beta > 0$. Furthermore, dimension is always assumed to be $n \in \{2,3\}$.

For a domain $\Omega \subset \mathbb{R}^n$, a Banach space X, and $1 \le p \le \infty$ in the sequel $L^p(\Omega, X)$ denotes the standard Bochner-Lebesgue space with norm

$$\|u\|_{L^p(X)} = \left(\int_\Omega \|u(x)\|_X^p\,dx\right)^{1/p},$$

if $1 \le p < \infty$ and $\|u\|_{L^\infty(X)} = \operatorname{ess\,sup}_{x\in\Omega}\|u\|_X$ if $p = \infty$. In case that $\Omega = X = \mathbb{R}^n$ its subspace of solenoidal functions is denoted by

$$L^p_\sigma(\mathbb{R}^n) := \{v \in L^p(\mathbb{R}^n);\ \operatorname{div} v = 0\}.$$

Note that for $1 < p < \infty$ we have the Helmholtz decomposition

$$L^p(\mathbb{R}^n) = L^p_\sigma(\mathbb{R}^n) \oplus G_p(\mathbb{R}^n)$$

with $G_p(\mathbb{R}^n) = \{\nabla p;\ p \in \mathscr{D}'(\mathbb{R}^n),\ \nabla p \in L^p(\mathbb{R}^n)\}$. The associated Helmholtz projector onto $L^p_\sigma(\mathbb{R}^n)$ is represented as

$$P = \mathscr{F}^{-1}\left(I - \frac{\xi\xi^T}{|\xi|^2}\right)\mathscr{F},$$

where $\mathscr{F}$ denotes the Fourier transformation and I the identity matrix in $\mathbb{R}^n$.

The symbol $W^{k,p}(\Omega, X)$, $k \in \mathbb{N}_0$, $1 \le p \le \infty$, stands for the standard Sobolev space of k-times differentiable functions in $L^p(\mathbb{R}^n, X)$. Its norm is given as

$$\|f\|_{W^{k,p}(X)} := \left(\sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^p(X)}^p \right)^{1/p}$$

with the usual modification if $p = \infty$. The fractional order Sobolev resp. Besov spaces are defined by complex resp. real interpolation as

$$W^{t,p}(\Omega, X) := \left[W^{k,p}(\Omega, X), W^{k+1,p}(\Omega, X) \right]_{t/k},$$
$$B_p^t(\Omega, X) := \left(W^{k,p}(\Omega, X), W^{k+1,p}(\Omega, X) \right)_{t/k,p}$$

for $t \in (0,k)$ and $k \in \mathbb{N}_0$. For $p = 2$ and $s \ge 0$ we use the notation $H^s(\Omega, X) := W^{s,2}(\Omega, X)$ and we frequently write $L^p(\Omega)$, $W^{s,p}(\Omega)$, and $B_p^s(\Omega)$ if $X = \mathbb{R}^n$. Also note that $H^s(\Omega) = W^{s,2}(\Omega) = B_2^s(\Omega)$, but that $W^{s,p}(\Omega, X) \neq B_p^s(\Omega, X)$ in general. Finally, $\mathscr{L}(X,Y)$ denotes the space of all bounded and linear operators from the space X into the space Y, we write $\mathscr{L}(X)$ if $X = Y$, and $\sigma(A)$ denotes the spectrum of a linear operator $A : D(A) \subset X \to X$.

3.1 Linear Theory

In this subsection we consider the linearized system

$$\begin{aligned} u_t + \lambda_0 (V \cdot \nabla) u + Mu - \Gamma_0 \Delta u + \Gamma_2 \Delta^2 u + \nabla q = f \quad &\text{in } (0,\infty) \times \mathbb{R}^n, \\ \operatorname{div} u = 0 \quad &\text{in } (0,\infty) \times \mathbb{R}^n, \\ u(0) = u_0 \ &\text{in } \mathbb{R}^n. \end{aligned} \tag{8}$$

Thanks to $\Gamma_2 > 0$ the operator

$$A_{SH} u := \Gamma_2 \Delta^2 u, \quad u \in D(A_{SH}) := W^{4,p}(\mathbb{R}^n) \cap L_\sigma^p(\mathbb{R}^n),$$

admits a bounded H^∞-calculus on $L_\sigma^p(\mathbb{R}^n)$ with H^∞-angle $\phi_{A_{SH}}^\infty = 0$ for $p \in (1,\infty)$. This follows as an easy consequence of Mikhlin's multiplier theorem, for instance. See e.g. [6, 19, 13] for an introduction to the notion of a bounded H^∞-calculus. Since every other term appearing in (8), more precisely the operator

$$Bu := \lambda_0 (V \cdot \nabla) u + PMu - \Gamma_0 \Delta u, \tag{9}$$

is of lower order, by a standard perturbation argument we immediately deduce

Lemma 1. *Let $1 < p < \infty$. There is an $\omega > 0$ such that the operator $\omega + A_{LF}$, where*

$$A_{LF} u := (A_{SH} + B) u, \quad u \in D(A_{LF}) := W^{4,p}(\mathbb{R}^n) \cap L_\sigma^p(\mathbb{R}^n), \tag{10}$$

admits a bounded H^∞-calculus on $L_\sigma^p(\mathbb{R}^n)$ with H^∞-angle $\phi_{\omega + A_{LF}}^\infty < \pi/2$.

As a consequence, cf. [6], $-A_{LF}$ is the generator of an analytic C_0-semigroup on $L^p_\sigma(\mathbb{R}^n)$ and it has maximal regularity:

Corollary 1. *Let $1 < p < \infty$, $T \in (0,\infty)$. For $f \in L^p((0,T), L^p_\sigma(\mathbb{R}^n))$ and $u_0 \in B_p^{4-4/p}(\mathbb{R}^n) \cap L^p_\sigma(\mathbb{R}^n)$ there exists a unique solution (u,q) of (8) satisfying*

$$\begin{aligned}&\|u\|_{W^{1,p}((0,T),L^p)} + \|u\|_{L^p((0,T),W^{4,p})} + \|\nabla q\|_{L^p((0,T),L^p)}\\ &\le C\left(\|f\|_{L^p((0,T),L^p)} + \|u_0\|_{B_p^{4-4/p}}\right)\end{aligned}$$

with $C > 0$ independent of u, q, f, u_0.

To obtain preciser information on the spectrum of the operator A_{LF} we apply Fourier transformation to (10) to the result that

$$\sigma_{A_{LF}}(\xi) = \mathscr{F} A_{LF} \mathscr{F}^{-1} = \Gamma_2|\xi|^4 + \Gamma_0|\xi|^2 + \sigma_P(\xi)M + i\lambda_0 V\cdot\xi, \quad \xi \in \mathbb{R}^n,$$

with $\sigma_P(\xi) = (1 - \xi\xi^t/|\xi|^2)$ the symbol of the Helmholtz projector P. We first consider the disordered state (2). We set $A_d := A_{LF}$ in this case. Then according to (6) the above expression takes the form

$$\sigma_{A_d}(\xi) = \Gamma_2|\xi|^4 + \Gamma_0|\xi|^2 + \alpha, \quad \xi \in \mathbb{R}^n.$$

Calculating the intersection points of the parabola in $s = |\xi|^2$ we obtain

$$s_\pm^2 = \frac{-\Gamma_0}{\Gamma_2}\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\alpha\Gamma_2}{\Gamma_0^2}}\right).$$

Consequently, if $\Gamma_0 < 0$ there is an unstable band of modes for $s^2 \in (s_-^2, s_+^2)$ provided this interval is nonempty. In this case the spectral bound $s(A_d) = \sup\{\mathrm{Re}\, z;\, z \in \sigma(A_d)\}$ of $-A_d$ is positive. Since it is well known that for analytic C_0-semigroups spectral bound of the generator and growth bound of the generated semigroup coincide [23], we deduce that $\exp(-tA_d)$ is exponentially unstable precisely if $\Gamma_0 < 0$ and $4\alpha < \Gamma_0^2/\Gamma_2$, or if $\Gamma_0 \ge 0$ and $\alpha < 0$. This leads to the following result.

Proposition 1. *Let $\Gamma_2 > 0$, $\beta > 0$, and $1 < p < \infty$. If $\Gamma_0 < 0$, then the disordered state (2) is linearly stable if and only if $4\alpha \ge \Gamma_0^2/\Gamma_2$. If $\Gamma_0 \ge 0$, then the disordered state (2) is stable if and only if $\alpha \ge 0$. To be precise, the semigroup $(\exp(-tA_d))_{t\ge 0}$ on $L^p_\sigma(\mathbb{R}^n)$ corresponding to the disordered state (2) is*

(1) exponentially stable if $\Gamma_0 < 0$ and $4\alpha > \Gamma_0^2/\Gamma_2$, or if $\Gamma_0 \ge 0$ and $\alpha > 0$;
(2) asymptotically stable if $\Gamma_0 < 0$ and $4\alpha = \Gamma_0^2/\Gamma_2$, or if $\Gamma_0 \ge 0$ and $\alpha = 0$;
(3) exponentially unstable if $\Gamma_0 < 0$ and $4\alpha < \Gamma_0^2/\Gamma_2$, or if $\Gamma_0 \ge 0$ and $\alpha < 0$.

Proof. It remains to prove (2), the other assertions are obvious by the discussion above. On the other hand, in the situation of (2) we see that $(\exp(-tA_d))_{t\ge 0}$ is a bounded analytic C_0-semigroup on $L^p_\sigma(\mathbb{R}^n)$, which are known to be asymptotically stable, see [23].

Next, we consider the manifold $B_{\alpha,\beta}$ of ordered polar states (3). In this case we set $A_o := A_{LF}$ and the symbol of this operator according to (7) reads as

$$\sigma_{A_o}(\xi) = \Gamma_2|\xi|^4 + \Gamma_0|\xi|^2 + 2\beta\sigma_P(\xi)VV^t + i\lambda_0 V\cdot\xi, \quad \xi\in\mathbb{R}^n,$$

with $V\in B_{\alpha,\beta}$. Note that the matrix $\sigma_P(\xi)VV^t$ is positive semidefinite and that zero is an eigenvalue by the fact that $V^t x = 0$ if $x\in\mathbb{R}^n\setminus\{0\}$ is perpendicular to V. Choosing $x,\xi\in\{V\}^\perp$ such that $|x|=1$ and that $|\xi|$ is small enough, we can always achieve that

$$x^t\sigma_{A_o}(\xi)x = \Gamma_2|\xi|^4 + \Gamma_0|\xi|^2 < 0,$$

provided that $\Gamma_0 < 0$. Thus, here we obtain the following result.

Proposition 2. *Let $1<p<\infty$, $\Gamma_2>0$, $\beta>0$, and $\alpha<0$. The ordered polar state (3) is linearly stable if and only if $\Gamma_0\geq 0$. To be precise, the semigroup $(\exp(-tA_o))_{t\geq 0}$ corresponding to the ordered state (3) is*

(1) exponentially unstable on $L^p_\sigma(\mathbb{R}^n)$ if $\Gamma_0<0$;
(2) asymptotically stable on $L^2_\sigma(\mathbb{R}^n)$ if $\Gamma_0\geq 0$.

Proof. Assertion (1) is clear. To see (2) first observe that due to the occurrence of the term $i\lambda_0 V\cdot\xi$, $(e^{-A_o t})_{t\geq 0}$ is not a bounded analytic semigroup. Hence we cannot argue as for A_d to deduce asymptotic stability. Instead, for (2) we restrict ourselves to the case $p=2$ and proceed as in [15]: For $v_0\in H^1(\mathbb{R}^n)\cap L^2_\sigma(\mathbb{R}^n)$, $v(t) := \exp(-tA_o)v_0$ solves

$$v_t(t) + \Gamma_2\Delta^2 v(t) - \Gamma_0\Delta v(t) + \lambda_0(V\cdot\nabla)v(t) + 2\beta PVV^t v(t) = 0.$$

Multiplication in $L^2(\mathbb{R}^n)$ with $v(t)$ and integration from $t=0$ to T yields

$$\|v(T)\|^2_{L^2} + 2\int_0^T \big(\Gamma_2\|\Delta v(t)\|^2_{L^2} + \Gamma_0\|\nabla v(t)\|^2_{L^2} + 2\beta\|V\cdot v(t)\|^2_{L^2}\big)\,dt = \|v_0\|^2_{L^2}$$

for every $T>0$. This has two consequences: First, the semigroup $\exp(-tA_o)$ is contractive since it is strongly continuous and since $H^1(\mathbb{R}^n)\cap L^2_\sigma(\mathbb{R}^n)$ is dense in $L^2_\sigma(\mathbb{R}^n)$, and second, we have

$$\int_0^\infty \|\Delta v(t)\|^2_{L^2}dt < \infty. \tag{11}$$

If $v_0\in H^4(\mathbb{R}^n)$, using the contractiveness we obtain

$$\begin{aligned}\left|\frac{d}{dt}\|\Delta v(t)\|^2_{L^2}\right| &= |2\langle\Delta v(t),\Delta v_t(t)\rangle| \\ &= \big|-2\langle\Delta^2\exp(-tA_o)v_0, A_o v(t)\rangle\big| \leq C\|v_0\|^2_{H^4},\end{aligned}$$

i.e. $\|\Delta v(\cdot)\|^2_{L^2}\in BC^1(0,\infty)$. Together with (11) this implies

$$\lim_{t\to\infty}\|\exp(-tA_o)\Delta v_0\|_{L^2} = \lim_{t\to\infty}\|\Delta v(t)\|_{L^2} = 0. \tag{12}$$

In order to prove asymptotic stability we have to show that for every $u_0 \in L^2_\sigma(\mathbb{R}^n)$, $u(t) := \exp(-tA_o)u_0$ satisfies

$$\lim_{t\to\infty} \|u(t)\|_{L^2} = 0.$$

Considering that $\{\Delta w; w \in H^4(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)\}$ is dense in $L^2_\sigma(\mathbb{R}^n)$ we can always find $v_0 \in H^4(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)$ with $\|u_0 - \Delta v_0\|_{L^2}$ arbitrary small. Making once more use of the contractiveness of the semigroup we obtain

$$\begin{aligned}\|u(t)\|_{L^2} &\le \|\exp(-tA_o)u_0 - \exp(-tA_o)\Delta v_0\|_{L^2} + \|\exp(-tA_o)\Delta v_0\|_{L^2}\\ &\le \|u_0 - \Delta v_0\|_{L^2} + \|\exp(-tA_o)\Delta v_0\|_{L^2},\end{aligned}$$

and (12) yields the asymptotic stability.

3.2 Local and global strong solvability

We first consider local-in-time wellposedness. For $T > 0$ we define relevant function spaces as

$$\begin{aligned}&\mathbb{E}_T := W^{1,p}((0,T), L^p_\sigma(\mathbb{R}^n)) \cap L^p((0,T), W^{4,p}(\mathbb{R}^n)),\\ &\mathbb{F}^1_T := L^p((0,T), L^p_\sigma(\mathbb{R}^n)), \quad \mathbb{F}^2 := B^{4-4/p}_p(\mathbb{R}^n),\\ &\mathbb{F}_T := \mathbb{F}^1_T \times \mathbb{F}^2,\end{aligned}$$

and the linear operator

$$L : \mathbb{E}_T \to \mathbb{F}_T, \quad Lu := (\partial_t u + A_{LF} u, u(0)).$$

If we also set

$$H(u) := \beta P|u|^2 u + \lambda_0 P(u\cdot\nabla)u - PN(u) \tag{13}$$

and

$$F(u) := Lu + (H(u), 0), \tag{14}$$

then the full system (5) is rephrased as

$$F(u) = (f, u_0).$$

Lemma 2. *Let $p > (4+n)/4$. We have $H \in C^1(\mathbb{E}_T, \mathbb{F}_T)$ and its Fréchet derivative is represented as*

$$DH(v)u = P\sum_{|\alpha|\le 1} b_\alpha \partial^\alpha u + \lambda_0 P(u\cdot\nabla)v, \quad u, v \in \mathbb{E}, \tag{15}$$

with matrices $b_\alpha = b_\alpha(v) \in L^\infty((0,T)\times\mathbb{R}^n, \mathbb{R}^{n\times n})$.

Proof. First observe that by [1, Proposition 1.4.2] we have

$$\mathbb{E}_T \hookrightarrow L^\infty((0,T), B_p^{4-4/p}(\mathbb{R}^n)). \tag{16}$$

The assumption $p > (4+n)/4$ in combination with the Sobolev embedding yields

$$\mathbb{E}_T \hookrightarrow L^\infty((0,T)\times\mathbb{R}^n). \tag{17}$$

Utilizing this fact we obtain

$$\begin{aligned}
\|(u\cdot\nabla)u\|_{\mathbb{F}_T^1} &\le C\|u\|_\infty\|\nabla u\|_{\mathbb{F}_T^1} \le C\|u\|_{\mathbb{E}_T}^2,\\
\||u|^2u\|_{\mathbb{F}_T^1} &\le C\||u|^2\|_\infty\|u\|_{\mathbb{F}_T^1} \le C\|u\|_{\mathbb{E}_T}^3,\\
\|N(u)\|_{\mathbb{F}_T^1} &\le C\|u\|_\infty\|u\|_{\mathbb{F}_T^1} \le C\|u\|_{\mathbb{E}_T}^2,
\end{aligned}$$

hence $H : \mathbb{E}_T \to \mathbb{F}_T$. By the fact that H consists of bi- and trilinear terms it is obvious that $H \in C^1(\mathbb{E}_T, \mathbb{F}_T)$ (even $H \in C^\infty(\mathbb{E}_T, \mathbb{F}_T)$). The Fréchet derivative reads as

$$\begin{aligned}
DH(v)u = {}&\beta P|v|^2u + 2\beta P(u\cdot v)v + \lambda_0 P(u\cdot\nabla)v\\
&+ \lambda_0 P(v\cdot\nabla)u - 2P\sum_{i,k=1}^{n} a_{jk}(u^jv^k + u^kv^j).
\end{aligned}$$

From this and (17) representation (15) easily follows.

Remark 1. Note that the lower bound $p > (4+n)/4$ is not optimal. But, since here we are mainly interested in an L^2-approach for dimension $n = 2,3$, it is sufficient for our purposes.

Lemma 3. *Let $p > (4+n)/4$, $T \in (0,\infty)$, and $v \in \mathbb{E}_T$. Then we have*

$$L + (DH(v), 0) \in \mathscr{L}_{is}(\mathbb{E}_T, \mathbb{F}_T).$$

Proof. By employing representation (15) for $B(t) := DH(v(t))$ we will show that $B(\cdot)$ is a lower order perturbation of L. To this end, observe that $p > (4+n)/4$ yields

$$W^{3,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$$

and thanks to (16) also $v \in L^\infty((0,T), W^{1,p}(\mathbb{R}^n))$. This implies

$$\begin{aligned}
\|(u\cdot\nabla)v(t)\|_{L^p(\mathbb{R}^n)} &\le \|\nabla v(t)\|_p\|u\|_\infty\\
&\le C\|v\|_{L^\infty((0,T),W^{1,p})}\frac{1}{\mu^{1/4}}\|(\mu + A_{LF})u\|_p
\end{aligned}$$

for all $u \in D(A_{LF})$ and all $\mu > \mu_0$ with $\mu_0 > 0$ large enough. We estimate the second term in (15) as

$$\begin{aligned}\|\sum_{|\alpha|\leq 1} b_\alpha \partial^\alpha u\|_{L^p(\mathbb{R}^n)} &\leq C\|v\|_\infty \|\nabla u\|_p \\ &\leq C\|v\|_\infty \frac{1}{\mu^{3/4}} \|(\mu + A_{LF})u\|_p\end{aligned}$$

for all $u \in D(A_{LF})$ and all $\mu > \mu_0$. This shows that

$$\|B(t)u\|_p \leq \frac{C(v)}{\mu^{1/4}} \|(\mu + A_{LF})u\|_p \quad (u \in D(A_{LF}),\ \mu > \mu_0).$$

Thus, choosing μ large enough and due to Corollary 1, we can apply [26, Theorem 2.5] to the result that

$$L + (\mu + DH(v), 0) \in \mathscr{L}_{is}(\mathbb{E}_T, \mathbb{F}_T).$$

Since $L + DH(v)$ is linear, we can remove the shift $\mu > 0$ and the assertion follows.

Appealing to the local inverse theorem we can now prove the following result.

Theorem 1 (local wellposedness). *Let $\Gamma_2, \beta > 0$, $\Gamma_0, \alpha \in \mathbb{R}$, and $p > (4+n)/4$. For every $u_0 \in B_p^{4-4/p}(\mathbb{R}^n) \cap L^p_\sigma(\mathbb{R}^n)$ and $f \in L^p((0,T), L^p_\sigma(\mathbb{R}^n))$ there exists a $T > 0$ and a unique solution (u,q) of (5) such that*

$$\begin{aligned} u &\in W^{1,p}((0,T), L^p_\sigma(\mathbb{R}^n)) \cap L^p((0,T), W^{4,p}(\mathbb{R}^n)), \\ \nabla q &\in L^p((0,T), L^p(\mathbb{R}^n)). \end{aligned}$$

Proof. We fix $(f, u_0) \in \mathbb{F}_T$ and define a reference solution as

$$u^* := L^{-1}(f, u_0) \in \mathbb{E}_T.$$

For the Fréchet derivative of the nonlinear operator $F \in C^1(\mathbb{E}_T, \mathbb{F}_T)$ given in (14) we obtain in view of Lemma 3 that

$$DF(u^*) = L + DH(u^*) \in \mathscr{L}_{is}(\mathbb{E}_T, \mathbb{F}_T).$$

Hence the local inverse theorem yields neighborhoods $U \subset \mathbb{E}_T$ of u^* and $V \subset \mathbb{F}_T$ of $F(u^*)$ such that $F : U \to V$ is bijective.

Now, taking $T' > 0$ small enough, we find a solution. To see this, for $0 < T' < T$ we define $f_{T'} \in \mathbb{F}_T^1$ by

$$f_{T'}(t) := \begin{cases} f(t), & t \in (0, T') \\ f(t) + H(u^*)(t), & t \in [T', T). \end{cases}$$

The continuity of the integral implies

$$f_{T'} \xrightarrow{T' \to 0} f + H(u^*) \text{ in } \mathbb{F}_T^1.$$

Thus, since $F(u^*) = (f + H(u^*), u_0)$, there is $T' > 0$ with $(f_{T'}, u_0) \in V$. The unique function $u \in U$ with $F(u) = (f_{T'}, u_0)$ satisfies $F(u) = (f, u_0)$ on $(0, T')$. Thus, recovering the pressure via

$$\nabla q := -(I-P)\left[\lambda_0 [u\cdot\nabla]u + (M+\beta|u|^2)u - N(u)\right] \in L^p((0,T), L^p(\mathbb{R}^n))$$

we find that $(u,q)|_{(0,T')}$ is a strong solution to (5).

The just constructed local solution extends to a global one, at least if $p = 2$.

Theorem 2 (global wellposedness). *Let $\Gamma_2, \beta > 0$, $\Gamma_0, \alpha \in \mathbb{R}$, $T \in (0,\infty)$. For every $u_0 \in H^2(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)$ and $f \in L^2((0,T), L^2_\sigma(\mathbb{R}^n))$ there exists a unique solution (u,q) of (5) such that*

$$\begin{aligned} u &\in H^1((0,T), L^2_\sigma(\mathbb{R}^n)) \cap L^2((0,T), H^4(\mathbb{R}^n)),\\ \nabla q &\in L^2((0,T), L^2(\mathbb{R}^n)). \end{aligned}$$

Proof. We derive a priori bounds in the strong class which will give the result. To this end, we multiply (5) with u and integrate over $(0,t)\times\mathbb{R}^n$. This yields

$$\begin{aligned} \frac{1}{2}\|u(t)\|_2^2 + \Gamma_2\int_0^t \|\Delta u\|_2^2\,ds + \beta\int_0^t \|u\|_4^4\,ds &= \frac{1}{2}\|u_0\|_2^2 + \int_0^t\int_{\mathbb{R}^n} fu\,dxds\\ + \Gamma_0\int_0^t\int_{\mathbb{R}^n} u\Delta u\,dxds - \int_0^t\int_{\mathbb{R}^n} uMu\,dxds &+ \int_0^t\int_{\mathbb{R}^n} uN(u)\,dxds \end{aligned}$$

for $t \in (0,T)$. By applying Cauchy-Schwarz' and Young's inequality we can estimate as

$$\int_0^t\int_{\mathbb{R}^n} u\Delta u\,dxds \le \frac{\Gamma_2}{2|\Gamma_0|}\|\Delta u\|^2_{L^2((0,t),L^2)} + C\int_0^t \|u\|_2^2\,dt$$

and

$$\int_0^t\int_{\mathbb{R}^n} uN(u)\,dxds \le \frac{\beta}{2}\|u\|^4_{L^4((0,t),L^4)} + C\int_0^t \|u\|_2^2\,dt. \tag{18}$$

Plugging this into the above equality we arrive at

$$\begin{aligned} &\|u(t)\|_2^2 + \|u\|^2_{L^2((0,t),H^2)} + \|u\|^4_{L^4((0,t),L^4)}\\ &\quad\le C\left(\|u_0\|_2^2 + \|f\|^2_{L^2((0,t),L^2)}\right) + C\int_0^t \|u(s)\|_2^2\,ds \quad (t\in(0,T)). \end{aligned}$$

Hence, Gronwall's lemma yields

$$\begin{aligned} &\|u\|^2_{L^\infty((0,T),L^2)} + \|u\|^2_{L^2((0,T),H^2)} + \|u\|^4_{L^4((0,T),L^4)}\\ &\quad\le C(1+Te^{\omega T})\left(\|u_0\|_2^2 + \|f\|^2_{L^2((0,T),L^2)}\right) \end{aligned} \tag{19}$$

for some $C, \omega > 0$.

Next we multiply (5) with $-\Delta u$ and obtain by utilizing integration by parts

$$\begin{aligned}
&\frac{1}{2}\|\nabla u(t)\|_2^2 + \Gamma_2 \int_0^t \|\nabla \Delta u\|_2^2\, ds + \beta \int_0^t \int_{\mathbb{R}^n} (\nabla |u|^2 u) \nabla u\, dx ds \\
&= \frac{1}{2}\|\nabla u_0\|_2^2 - \int_0^t \int_{\mathbb{R}^n} f \Delta u\, dx ds - \Gamma_0 \|\Delta u\|^2_{L^2((0,t),L^2)} \\
&\quad - \int_0^t \int_{\mathbb{R}^n} (\nabla u) M \nabla u\, dx ds - \int_0^t \int_{\mathbb{R}^n} (\Delta u) N(u)\, dx ds \\
&\quad - \lambda_0 \int_0^t \int_{\mathbb{R}^n} [(V+u)\cdot \nabla] u \Delta u\, dx ds.
\end{aligned} \tag{20}$$

Concerning the third term on the left hand side we calculate

$$\begin{aligned}
(\nabla |u|^2 u) \nabla u &= \sum_{j,k,\ell=1}^{n} (\partial_k u^j) \partial_k (u^\ell)^2 u^j \\
&= \sum_{j,k,\ell=1}^{n} (u^\ell)^2 (\partial_k u^j)^2 + 2 \sum_{k=1}^{n} (u \cdot \partial_k u)^2.
\end{aligned}$$

This shows that this term is non-negative, hence it drops out. For the fifth term on the right hand side analogously to (18) we obtain

$$\int_0^t \int_{\mathbb{R}^n} (\Delta u) N(u)\, dx ds \le C \left(\|u\|^4_{L^4((0,t),L^4)} + \int_0^t \|\Delta u\|_2^2\, ds \right),$$

whereas the last term on the right hand side can be estimated utilizing integration by parts and $\operatorname{div}(V+u) = 0$ as

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} [(V+u)\cdot \nabla] u \Delta u\, dx ds \\
&\le C \left(\|u\|^2_{L^2((0,t),L^2)} + \|u\|^4_{L^4((0,t),L^4)} \right) + \frac{\Gamma_2}{2|\lambda_0|} \int_0^t \|\nabla \Delta u\|_2^2\, dt.
\end{aligned}$$

Plugging this into (20) yields in combination with (19) that

$$\begin{aligned}
&\|u\|^2_{L^\infty((0,T),H^1)} + \|u\|^2_{L^2((0,T),H^3)} + \|u\|^4_{L^4((0,T),L^4)} \\
&\quad \le C(1 + T e^{\omega T}) \left(\|u_0\|^2_{H^1} + \|f\|^2_{L^2((0,T),L^2)} \right).
\end{aligned} \tag{21}$$

In the third step we multiply with $\Delta^2 u$ to obtain

$$\begin{aligned}
&\frac{1}{2}\|\Delta u(t)\|_2^2 + \Gamma_2 \int_0^t \|\Delta^2 u\|_2^2\, ds \\
&= \frac{1}{2}\|\Delta u_0\|_2^2 + \int_0^t \int_{\mathbb{R}^n} f \Delta^2 u\, dx ds - \Gamma_0 \|\Delta \nabla u\|^2_{L^2((0,t),L^2)} \\
&\quad - \int_0^t \int_{\mathbb{R}^n} (\Delta u) M \Delta u\, dx ds + \int_0^t \int_{\mathbb{R}^n} (\Delta^2 u) N(u)\, dx ds \\
&\quad - \lambda_0 \int_0^t \int_{\mathbb{R}^n} [(V+u)\cdot \nabla] u \Delta^2 u\, dx ds - \beta \int_0^t \int_{\mathbb{R}^n} |u|^2 u \Delta^2 u\, dx ds.
\end{aligned} \tag{22}$$

It is enough to focus on the last two terms on the right hand side. The remaining terms can be handled very similar as before. The first one of those two terms can be controlled by utilizing the estimate

$$\begin{aligned}\int_0^t \|(u\cdot\nabla)u\|_2^2\,ds &\le C\int_0^t \|u\|_4^2\|\nabla u\|_4^2\,ds\\ &\le C\|u\|_{L^\infty(H^1)}^2\int_0^t \|u\|_{H^2}^2\,ds\\ &\le C(1+Te^{\omega T})^2\left(\|u_0\|_{H^1}^2+\|f\|_{L^2((0,T),L^2)}^2\right)^2,\end{aligned}$$

which is valid thanks to (21) and the Sobolev embedding $H^1(\mathbb{R}^n)\hookrightarrow L^4(\mathbb{R}^n)$.

For the last term, using complex interpolation [32] we obtain

$$L^\infty((0,T),H^1(\mathbb{R}^n))\cap L^2((0,T),H^3(\mathbb{R}^n))\hookrightarrow L^{2/s}((0,T),H^{2s+1}(\mathbb{R}^n))$$

for $s\in[0,1]$. Taking into account $n\le 3$ the Sobolev embedding yields

$$H^{5/3}\hookrightarrow L^6(\mathbb{R}^n). \tag{23}$$

Hence, by setting $s=1/3$ we obtain

$$L^\infty((0,T),H^1(\mathbb{R}^n))\cap L^2((0,T),H^3(\mathbb{R}^n))\hookrightarrow L^6((0,T),L^6(\mathbb{R}^n)).$$

On the other hand, from Cauchy-Schwarz' and Young's inequality we see that

$$\left|\int_0^t\int_{\mathbb{R}^n}|u|^2u\Delta^2u\,dx\,ds\right|\le C\|u\|_{L^6((0,t),L^6)}^6+\frac{\Gamma_2}{2\beta}\|\Delta^2u\|_{L^2((0,t),L^2)}^2.$$

Here the second term on the right hand side is absorbed by the left hand side of (22) and the first term is again controlled by estimate (21). Summarizing, we arrive at

$$\begin{aligned}&\|u\|_{L^\infty((0,T),H^2)}^2+\|u\|_{L^2((0,T),H^4)}^2\\ &\quad\le C(1+Te^{\omega T})^3\left(\|u_0\|_{H^1}^2+\|f\|_{L^2((0,T),L^2)}^2\right)^3.\end{aligned}$$

Using equations (5) it is straight forward to derive similar bounds for the quantities $\|u\|_{H^1((0,T),L^2)}$ and $\|\nabla q\|_{L^2((0,T),L^2)}$ as well. Thus the assertion is proved.

3.3 Nonlinear (In-) Stability

Most of the outcome on linear (in-) stability in the L^2-setting transfers to the corresponding nonlinear situation. For the transfer of the stability results we apply energy methods and for the transfer of the results on instability we employ Henry's instability theorem [14, Corollary 5.1.6], which we reformulate suitably for our purposes.

Proposition 3. *Let $-A$ be the generator of a holomorphic C_0-semigroup in a Banach space X and let $f\colon U \to X$, where U is an open neighborhood in $X^\gamma := D(A^\gamma)$ for some $\gamma \in (0,1)$, be locally Lipschitz. Let $x_0 \in D(A) \cap U$ be an equilibrium point of*

$$\dot{w}(t) + Aw(t) = f(w(t)), \tag{24}$$

i.e. $Ax_0 = f(x_0)$. Suppose

$$f(x_0+z) = f(x_0) + Bz + g(z), \quad g(0) = 0,$$
$$\|g(z)\| = O(\|z\|^s_{X^\gamma}), \quad \text{as } z \to 0 \text{ in } X^\gamma,$$

for some $s > 1$, $B \in \mathscr{L}(X^\gamma, X)$, and $\sigma(-A+B) \cap \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\} \neq \emptyset$. Then x_0 is nonlinearly unstable in the following sense: there is a constant $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exists $x \in X^\gamma$ with $\|x - x_0\|_\gamma < \delta$ such that there is some finite time $t_0 > 0$ with

$$\|w(t_0,x) - x_0\|_{X^\gamma} \geq \varepsilon_0,$$

where $w(\cdot,x)$ denotes the solution of (24) with initial value $w(0,x) = x$.

For instability also the following lemma will be helpful.

Lemma 4. *Let the nonlinearity H be given as in (13). Then for every $\sigma \geq 5/4$ we have $H \in C^1(H^\sigma(\mathbb{R}^n), L^2_\sigma(\mathbb{R}^n))$ and*

$$\|H(u)\|_2 \leq C\|u\|^2_{H^\sigma} \quad (\|u\|_{H^\sigma} \leq 1).$$

Proof. Employing Hölder's inequality we obtain

$$\|(u\cdot\nabla)u\|_2 \leq \|u\|_p \|\nabla u\|_q$$

for $1/p + 1/q = 1/2$. It is easily checked that the Sobolev embeddings $H^\gamma(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ and $H^{\gamma-1}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ are sharp for $p = 12$, $q = 12/5$, and $\gamma = 5/4$. By the fact that

$$\||u|^2 u\|_2 = \|u\|^3_6 \leq C\|u\|^3_{H^1}$$

and since H consists of bi- and trilinear terms the assertion follows.

Remark 2. It is clear that due to better Sobolev embeddings the lower bound on σ can be improved if $n = 2$.

As before, first we consider the disordered state (2).

Theorem 3. *Let $\Gamma_2 > 0$ and $\beta > 0$. Then the disordered state (2) is nonlinearly*

(1) (globally) exponentially stable in $L^2_\sigma(\mathbb{R}^n)$ if $\Gamma_0 < 0$ and $4\alpha > \Gamma_0^2/\Gamma_2$, or if $\Gamma_0 \geq 0$ and $\alpha > 0$;
(2) stable in $L^2_\sigma(\mathbb{R}^n)$ if $\Gamma_0 < 0$ and $4\alpha = \Gamma_0^2/\Gamma_2$, or if $\Gamma_0 \geq 0$ and $\alpha = 0$;
(3) unstable in $H^\gamma(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)$ for $\gamma \in [5/16, 1)$ if $\Gamma_0 < 0$ and $4\alpha < \Gamma_0^2/\Gamma_2$, or if $\Gamma_0 \geq 0$ and $\alpha < 0$.

Proof. Suppose u and q with regularity as in Proposition 2 solve the nonlinear system (5) corresponding to the disordered state (2), that is

$$u_t + \Gamma_2 \Delta^2 u - \Gamma_0 \Delta u + \lambda_0 (u \cdot \nabla) u + (\alpha + \beta |u|^2) u + \nabla q = 0.$$

Testing this equation with u we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \Gamma_2 \|\Delta u\|_2^2 + \Gamma_0 \|\nabla u\|_2^2 + \alpha \|u\|_2^2 + \beta \|u\|_4^4 = 0.$$

If $\Gamma_0 \geq 0$ and $\alpha \geq 0$, all coefficients are nonnegative and we deduce

$$\frac{d}{dt}\|u\|_2^2 \leq -2\alpha \|u\|_2^2,$$

which yields

$$\|u(t)\|_2^2 \leq e^{-2\alpha t}\|u_0\|_2^2 \qquad (t \geq 0),$$

i.e. stability if $\alpha = 0$ and exponential stability if $\alpha > 0$.

If $\Gamma_0 < 0$, we use the Plancherel theorem, Hölder's inequality, and Young's inequality with ε to estimate the term

$$\begin{aligned}\|\nabla u\|_2^2 &= \int_{\mathbb{R}^n} |\xi|^2 |\hat{u}(\xi)|^2 d\xi \leq \||\xi|^2 \hat{u}(\xi)\|_2 \|\hat{u}\|_2 \\ &= \|\Delta u\|_2 \|u\|_2 \leq \frac{\varepsilon^2}{2}\|\Delta u\|_2^2 + \frac{1}{2\varepsilon^2}\|u\|_2^2.\end{aligned}$$

The $\varepsilon^2/2$-term can be absorbed by the Γ_2-term if we choose $\varepsilon^2 = 2\Gamma_2/|\Gamma_0|$. Dropping the β-term as well we are left with

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \alpha \|u\|_2^2 \leq \frac{\Gamma_0^2}{4\Gamma_2}\|u\|_2^2.$$

As before, this implies stability if $4\alpha = \Gamma_0^2/\Gamma_2$ and exponential stability if $4\alpha > \Gamma_0^2/\Gamma_2$. Thus assertions (1) and (2) are proved.

To see instability we apply Proposition 3. In our situation we have $x_0 = 0$, $B = 0$, $x = u$, $A = A_d$, and $f(u) = H(u)$. From this we also see that $H(0) = 0$, i.e., that $g = f = H$ in Henry's notation. Note that Proposition 1(3) under the assumption (3) above implies that $\sigma(-A_d) \cap \{z \in \mathbb{C} : \ \mathrm{Re}\, z > 0\} \neq \emptyset$. Next, the fact that $\omega + A_d$ admits a bounded H^∞-calculus for some $\omega > 0$ (Lemma 1) yields

$$D(A_d^\gamma) = \left[L_\sigma^2(\mathbb{R}^n), D(A_d)\right]_\gamma = H^{4\gamma}(\mathbb{R}^n) \quad (\gamma \in [0,1]),$$

where $[\cdot,\cdot]_s$ denotes the complex interpolation space, cf. [32]. Lemma 4 then implies that the assertions of Proposition 3 are fulfilled for $\gamma \in [5/16, 1)$ and $s = 2$. Hence the disordered state is unstable.

Finally we consider instability for the ordered polar state.

Theorem 4. *Let $\Gamma_2, \beta > 0$ and $\Gamma_0, \alpha < 0$. Then the ordered polar state (3) is nonlinearly unstable in $H^\gamma(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)$ for $\gamma \in [5/16, 1)$.*

Proof. Based on Lemma 4 and Proposition 2(1) the proof is analogous to the proof of Theorem 3(3).

Remark 3. We have seen in Theorem 3 that for the disordered steady state the results on linear (in-) stability in principle completely transfer to the nonlinear situation. Note that for the time being it is not clear if for $\Gamma_0 \geq 0$ the linear stability for the ordered polar state given by Proposition 2(2) transfers to the nonlinear situation as well. Proceeding as for the disordered state, i.e., employing energy methods, it appears that for the ordered state 'disturbing' terms on the right hand side can not easily be absorbed by the 'good' terms on the left hand side by just applying Young, Sobolev and Hölder. It seems that here a refined analysis is required which, however, is left as a future challenge.

4 Conclusions and future developments

In conclusion, we have shown that the model proposed by Wensink et al. in [33] gives rise to a mathematically wellposed system which reflects the asymptotic behavior observed in simulations and experiments. In detail we have proved:

(i) existence of a unique local-in-time solution for initial data in $B_p^{4-4/p}(\mathbb{R}^n) \cap L^p_\sigma(\mathbb{R}^n)$;
(ii) existence of a unique global strong solution for arbitrary initial data in $H^2(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)$;
(iii) results on stability and instability of the ordered and the disordered steady states in the L^2-setting depending on the values of the occurring physically relevant parameters.

Note that (ii) is in contrast to the (mathematical) situation of the classical Navier-Stokes system. The fact that we can prove existence of a unique global strong solution here, of course essentially relies on the presence of the fourth order term in (1) which provides sufficient regularity.

The intention of this note is to give an analytical approach in the L^2-setting which serves as a first step for further thorough examinations in several directions. Future work should address the following problems: first of all, it should also be mentioned that the generalized Navier-Stokes framework was augmented even further by including further higher derivatives in the velocities entering into the stress tensor, see the recent work by Slomka and Dunkel [28]. We anticipate that our analysis can also be performed in this more general case provided the highest order term has the correct stabilizing sign.

Second, another more complex problem is that of boundary conditions for the fluid velocity field. This is important to take into account walls and obstacles which confine the bacterial flow. In fact, recent experiments with bacterial turbulence were performed with mobile wedge-like obstacles or carriers [16, 17, 18] which were powered by activity or in static meso-structured environments [36] where there is an interesting competition between the geometric structure of the boundaries and the swirling. Recent observations have also addressed spheres as passive additives to steer bacterial turbulence. For all these interesting set-ups the mathematical analysis should include boundary conditions. The latter are typically stick or slip or involve a finite slipping length. Therefore the wellposedness and stability of the generalized Navier-Stokes equation in nontrivial boundaries should be addressed in future studies.

Third, some bacteria move on the surface of an emulsion droplet, which has motivated recent studies of active particles on a compact manifold, such as a sphere [27, 12, 20]. It would be interesting to generalize the hydrodynamic model (1) towards a nonplanar geometry and to prove wellposedness and stability for the problem on curved space.

References

1. Amann, H.: Linear and Quasilinear Parabolic Problems. Vol. I. Birkhäuser (1995)
2. Amann, H.: On the strong solvability of the Navier-Stokes equations. J. Math. Fluid Mech. **2**(1), 16–98 (2000)
3. Bechinger, C., di Leonardo, R., Löwen, H., Reichhardt, C., Volpe, G.: Active Brownian motion in complex and crowded environments. (submitted to Reviews of Modern Physics in 2016)
4. Bratanov, V., Jenko, F., Frey, E.: New class of turbulence in active fluids. PNAS **112**, 15,048–15,053 (2015)
5. Cannone, M.: Harmonic analysis tools for solving the incompressible Navier-Stokes equations. Handbook of Mathematical Fluid Dynamics. Vol. III. Elsevier/North Holland (2004)
6. Denk, R., Hieber, M., Prüss, J.: R-boundedness, Fourier multipliers and problems of elliptic and parabolic type. Mem. Amer. Math. Soc. **166**, viii+114 (2003)
7. Dunkel, J., Heidenreich, S., Bär, M., Goldstein, R.E.: Minimal continuum theories of structure formation in dense active fluids. New Journal of Physics **15**, 045,016 (2013)
8. Dunkel, J., Heidenreich, S., Drescher, K., Wensink, H.H., Bär, M., Goldstein, R.E.: Fluid dynamics of bacterial turbulence. Phys. Rev. Lett. **110**, 228,102 (2013)
9. Elgeti, J., Winkler, R.G., Gompper, G.: Physics of microswimmers–single particle motion and collective behavior: a review. Rep. Prog. Phys **78**, 056,601 (2015)
10. Emmerich, H., Löwen, H., Wittkowski, R., Gruhn, T., Tóth, G.I., Tegze, G., Gránásy, L.: Phase-field crystal models for condensed matter dynamics on atomic length and diffusive time scales: an overview. Advances in Physics **61**, 665–743 (2012)
11. Galdi, G.: An Introduction to the Mathematical Theory of the Navier-Stokes Equations, 2nd edn. Springer Verlag (2011)
12. Großmann, R., Peruani, F., Bär, M.: A geometric approach to self-propelled motion in isotropic and anisotropic environments. European Physical Journal Special Topics **224**, 1377–1394 (2015)
13. Haase, M.: The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications. Birkhäuser Verlag Basel (2006)
14. Henry, D.: Geometric Theory of Semilinear Parabolic Equations. Springer, Berlin-New York (1981)
15. Hess, M., Hieber, M., Mahalov, A., Saal, J.: Nonlinear stability of Ekman layers. Bull. Lond. Math. Soc. **42**(4), 691–706 (2010)

16. Kaiser, A., Peshkow, A., Sokolov, A., ten Hagen, B., Löwen, H., Aranson, I.S.: Transport powered by bacterial turbulence. Physical Review Letters **112**, 158,101 (1–5) (2014)
17. Kaiser, A., Sokolov, A., Aranson, I.S., Löwen, H.: Mechanisms of carrier transport induced by a microswimmer bath. IEEE Transactions on Nanobioscience **14**, 260–266 (2015)
18. Kaiser, A., Sokolov, A., Aranson, I.S., Löwen, H.: Motion of two micro-wedges in a turbulent bacterial bath. European Physical Journal: Special Topics **224**, 1275–1286 (2015)
19. Kunstmann, P., Weis, L.: Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems, and H^∞-functional calculus, *Lecture Notes in Math.*, vol. 1855. Springer (2003)
20. Li, W.: Collective motion of swarming agents evolving on a sphere manifold: A fundamental framework and characterization. Scientific Reports **5**, 13,603 (2015)
21. Marchetti, M.C., Joanny, J.F., Ramaswamy, S., Liverpool, T.B., Prost, J., Rao, M., Sima, A.: Hydrodynamics of soft active matter. Rev. Mod. Phys. **85**, 1143–1189 (2013)
22. Menzel, A.M.: Tuned, driven, and active soft matter. Physics Reports **554**, 1–45 (2015)
23. Nagel, R.: One-parameter Semigroups of Positive Operators, *Lecture Notes in Math.*, vol. 1184. Springer Verlag Berlin (1986)
24. Purcell, E.M.: Life at Low Reynolds Number. Am. J. Phys. **45**, 3–11 (1977)
25. Romanczuk, P., Bär, M., Ebeling, W., Lindner, B., Schimansky-Geier, L.: Active Brownian Particles. From Individual to Collective Stochastic Dynamics. Eur. Phys. J. Special Topics **202**, 1–162 (2012)
26. Saal, J.: Maximal regularity for the Stokes equations in non-cylindrical space-time domains. J. Math. Soc. Japan **58**(3), 617–641 (2006)
27. Sknepnek, R., Henkes, S.: Active swarms on a sphere. Phys. Rev. **91**, 022,306 (2015)
28. Slomka, J., Dunkel, J.: Generalized Navier-Stokes equations for active suspensions. European Physical Journal Special Topics **224**, 1349–1358 (2015)
29. Sohr, H.: The Navier-Stokes Equations. An Elementary Functional Analytic Approach. Birkhäuser (2001)
30. Swift, J., Hohenberg, P.C.: Hydrodynamic fluctuations at the convective instability. Phys. Rev. A **15**, 319–328 (1977)
31. Toner, J., Tu, Y.: Long-Range Order in a Two-Dimensional Dynamical XY Model: How Birds Fly Together. Phys. Rev. Lett. **75**, 4326–4329 (1995)
32. Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators. North Holland (1978)
33. Wensink, H.H., Dunkel, J., Heidenreich, S., Drescher, K., Goldstein, R.E., Löwen, H., Yeomans, J.M.: Meso-scale turbulence in living fluids. Proc. Natl. Acad. Sci. USA **109**, 14,308–14,313 (2012)
34. Wensink, H.H., Löwen, H.: Emergent states in dense systems of active rods: from swarming to turbulence. J. Phys.: Condens. Matter **24**, 464,130 (2012)
35. Wiegner, M.: The Navier-Stokes equations – a neverending challenge? Jahresber. Dtsch. Math.-Ver. **101**(1), 1–25 (1999)
36. Wioland, H., Woodhouse, F.G., Dunkel, J., Goldstein, R.E.: Ferromagnetic and antiferromagnetic order in bacterial vortex lattices. Nature Physics (2016)
37. Yeomans, J.M.: Active matter: Playful topology. Nature Materials **13**, 1004 (2014)